Introducing the arbitrary constant C_6, we obtain

$$\frac{\varkappa+1}{2n}\lambda\left[\lambda g_m-\frac{2}{\varkappa+1}(gf_m+fg_m)\right]-\left[\lambda(gf_m+fg_m-2gw_m)-\frac{1}{\varkappa+1}(4fgf_m+2f^2g_m-4fgw_m+(\varkappa-1)h_m)\right]=\frac{C_6}{\lambda^2}\qquad m=10n \tag{2.16}$$

The structure of integrals (2.15) and (2.16) is the same as that of integral (2.14), with the first term corresponding to the linearized integral of mass and the second to the linearized momentum integral [4].

Equation (1.9) of the divergent form yields for $\nu = 2$ and $\nu = 3$ the integral which defines flows with conservation of the moment of momentum of flow; such flows cannot be defined by expansions (2.3) for the shock wave propagating in a quiescent gas, and are not considered here.

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FLOWS OF A REACTING MIXTURE IN LAVAL NOZZLES UNDER CONDITIONS OF A QUASI-FROZEN PROCESS

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Flows of a chemically active gas mixture are considered in a small region of a Laval nozzle, where their mode changes from subsonic to supersonic (the frozen speed of sound is considered) are analyzed. Continuous solutions and solutions with shock waves are derived. Conditions of shock-free flows are obtained.

1. Continuous flows. We locate the system of coordinates (x, r) (cylindrical or Cartesian) at the nozzle axis of symmetry at the point where the stream velocity is equal to the frozen speed of sound, and combine the x-axis with the axis of symmetry. Under conditions of quasi-frozen process the system of equations of motion of a gas mixture is of the form [1]

$$v_x + v_x \frac{\partial v_x}{\partial x} - \frac{\partial v_r}{\partial r} = (\nu - 1)\frac{v_r}{r}, \quad \frac{\partial v_x}{\partial r} = \frac{\partial v_r}{\partial x} \tag{1.1}$$

(for plane-parallel flows $\nu = 1$ and for axisymmetric ones $\nu = 2$).

For system (1.1) at the nozzle axis of symmetry we formulate the following Cauchy problem:

$$v_x = A_1 x, \quad x < 0; \quad v_x = A_2 x, \ x > 0; \quad v_r = 0 \quad (A_1 > 0) \tag{1.2}$$

and seek in the considered flow region either continuous or discontinuous solutions of problem (1.1), (1.2).

Note that discontinuity of the derivative is admissible at point $x = 0, \ r = 0$. The magnitude of the latter determines the character of transition from subsonic to supersonic velocities.

The problem (1.1), (1.2) is invariant with respect to the continuous group of transformation of similitude

$$x \to \alpha x, \quad r \to \alpha^{1/2} r, \quad v_x \to \alpha x, \quad v_r = \alpha^{3/2} v_r$$

Hence its solution can be sought in the self-similar form

$$v_x = r^2 f(\xi), \quad v_r = r^3 g(\xi), \quad \xi = x / r^2 \tag{1.3}$$

The substitution of (1.3) into (1.1) yields equations that are satisfied by functions f and g. After the elimination of g in these, we obtain for f the second order equation

$$(f - 4\xi^2)\frac{d^2 f}{d\xi^2} + \left(\frac{df}{d\xi}\right)^2 + (2\nu\xi + 1)\frac{df}{d\xi} - 2\nu f = 0 \tag{1.4}$$

while g is determined by formula

$$g = \frac{1}{\nu + 2}\left[f\frac{df}{d\xi} - 4\xi^2\frac{df}{d\xi} + f(4\xi + 1)\right] \tag{1.5}$$

Equation (1.4) has a simple particular solution

$$f = A\xi + A(A + 1) / (2\nu) \tag{1.6}$$

where A is an arbitrary constant. Singular points of Eq. (1.4) correspond to singular characteristics which pass through the coordinate origin in the physical plane (x, r).

The solution of the problem is derived as follows. We denote by C_-° and C_+° the extreme left- and right-hand singular characteristics that pass through the coordinate origin, respectively. We divide the flow region into three parts, viz, region *1* lying to the left of C_-°, region *2* lying to the right of C_+°, and region *3* lying between the singular characteristics (Fig. 1). Integrals (1.6) with constant A equal A_1 or A_2 represent, respectively, solutions in regions *1* and *2*, with the characteristics $C_\pm^\circ$ of the form $x / r^2 = \xi_\pm^\circ = \text{const}$, where ξ_-° (ξ_+°) define the left-hand (right-hand) intersection point of parabola $f = 4\xi^2$ with the straight line (1.6) with $A = A_1$ $(A = A_2)$. The integral curves that correspond to actual physical flows can have at the intersection points with the parabola only two slopes

$$f_{\pm}' = -\nu\xi - {}^1/_2 \pm \sqrt{(\nu\xi + {}^1/_2)^2 + 8\nu\xi^2} \qquad (1.7)$$

determined by (1.4).

Behavior of these curves is qualitatively represented in Fig. 2. The coordinates of points a, b, c and d are, respectively: $(-{}^1/_4, \nu / 2)$; $(-{}^1/_4, -1)$; $(-1 / (\nu + 8), \nu / (\nu + 8))$; $(-1 / (\nu + 8), -8 / (\nu + 8))$.

Note that curves other than those defined by (1.6) can have points of inflection only on the parabola $f = 4\xi^2$. In fact, let at some point ξ_1 the quantity $f(\xi_1) \neq 4\xi^2$ and $d^2f / d\xi^2 = 0$, then Eq. (1.5) at point ξ_1 of the considered integral curve becomes

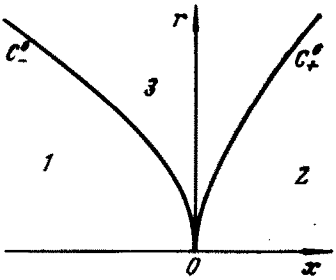

Fig. 1

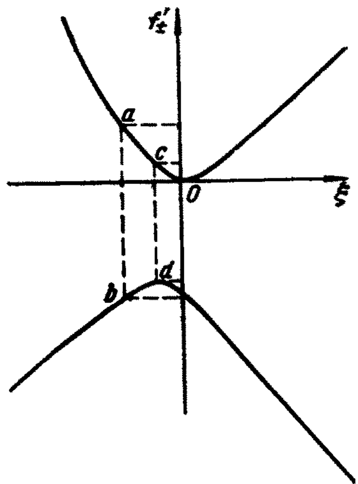

Fig. 2

$$(df / d\xi)^2 + (2\nu\xi + 1)\, df / d\xi - 2\nu f = 0 \qquad (1.8)$$

Hence by the theorem of existence and uniqueness we conclude that the considered integral curve is of the form (1.6). For region *3* the solution can be derived by numerical integration of (1.4) with the condition of its continuity along the singular characteristics C_-° and C_+° and of finiteness at the derivatives of these.

The analysis of finite expansions in the neighborhood of singular points $f = 4\xi^2$ shows that for $0 \leqslant A_1 \leqslant \nu / (\nu + 8)$ region *3* does not contain continuous solutions that are different from (1.6) with A determined by (1.7). For $A_1 > \nu / (\nu + 8)$ a beam of integral curves emanates from each point of intersection of the parabola with the straight line $f = A_1\xi + A_1 (A_1 + 1) / (2\nu)$. At that point the slope of integral curves A_1' determined by (1.7) for $\xi = \xi_-^\circ$ is negative. The unique solution which at that point has a positive slope is the straight line (1.6) with $A = A_1$.

Integral curves emanating from the parabola with a negative slope, which pass under the straight line (1.6) with $A = A_1'$ intersect the parabola for the second time at a negative slope. As previously shown, along such curves $d^2f / d\xi^2 < 0$ and $df / d\xi < A_1'$, and since the second intersection point of such curve with the parabola lies to the left of the second intersection point of the straight line (1.6) with $A = A_1'$ with the parabola, we conclude from the examination of Fig. 2 that the considered curve reaches the parabola $f = 4\xi^2$ at an infinite negative slope. Such solutions correspond to flows with infinite accelerations. Since this is physically impossible, either a shock wave must be generated in them or their pattern undergoes a complete change. It can be similarly shown that integral curves lying above the straight line (1.6) with $A = A_1'$ reach the right-hand branch of the parabola at a positive slope. Solution (1.6) with $A = A_1'$ is a limit one for continuous flows in region *3* when $A_1 > \nu/2$. The condition of absence of shock for plane flows is of the form

$$17A_1 / 8 - 5R / 2 + {}^1/_2 \leqslant A_2 \leqslant A_1 \qquad (1.9)$$

$$R = \sqrt{A_1^2 / 16 + A_1 (A_1 + 1) / 8} \quad (A_2 > 0)$$

For $A_2 < 0$ (which corresponds to flows with supersonic zones locked at the axis) it is of the form

$$-5A_1 / 4 - R - 1 \leqslant A_2 \leqslant -5A_1 / 4 + R - 1 \qquad (1.10)$$

For $A_1 \to \infty$ these conditions become ${}^1/_4 \leqslant A_2 / A_1 \leqslant 1$ and $-2 \leqslant A_2 / A_1 \leqslant -{}^1/_2$ which were derived in [2, 3] for an inert gas.

For $\nu / (\nu + 8) \leqslant A_1 \leqslant \nu / 2$ the limit solution for continuous flows in region *3* is a broken line consisting of a segment of the straight line (1.6) with $A = A_1'$ up to the second intersection with the parabola at point ξ_0 and with the straight line (1.6) with $A = A_1''$, where A_1'' is determined by (1.7) for $\xi = \xi_0$. This limit solution, unlike that for an inert gas, has three singular characterisitics (curve *4* in Fig. 4).

The conditions of absence of shock are readily derived in this case. For plane flows they are of the form

$$17A_1 / 8 - 5R / 2 + {}^1/_2 \leqslant A_2 \leqslant A_1 \quad (A_2 > 0) \qquad (1.11)$$

$$5A_1 / 4 - R - 1 \leqslant A_2 \leqslant -65A_1 / 4 + 21R / 4 - {}^9/_4 \qquad (1.12)$$

The conditions of absence of shock in the case of axisymmetric flows are similar. Results of calculations for $\nu = 1$ and $A_1 = 1$, and $A = 0.4$ are shown in Figs. 3 and 4, respectively.

2. Flows with shock waves. At the shock front specified implicitly by the equation $\varphi(x, r) = 0$, the following conditions must be satisfied:

$$(v_{x1}^2 - v_{x2}^2)\varphi_x / 2 - (v_{r1} - v_{r2})\varphi_r = 0, \qquad (2.1)$$

$$(v_{x1} - v_{x2})\varphi_r - (v_{r1} - v_{r2})\varphi_x = 0$$

where subscripts 1 and 2 relate the state ahead and behind the shock, respectively. Conditions (2.1) are exactly the same as at the shock front of a perfect gas [2]. By the Cemplen theorem $v_{x1} > v_{x2}$. Writing the equation of the shock front in the form $x / r^2 = \xi_s = \text{const}$ and allowing for (1.3) and (2.1), for $\nu = 1$ we obtain

$$f_1 + f_2 = 8\xi_s^2, \quad (df / d\xi)_1 + (df / d\xi)_2 + 2(10\xi_s + 1) = 0 \qquad (2.2)$$

Solutions behind the shock are defined by the integral (1.6) in which A and ξ_s are obtained from (2.2). Let us write (2.2) in a more convenient form

$$f_1 + \left[\left(\frac{df}{d\xi}\right)_1 + 2(10\xi_s + 1)\right]\left[\left(\frac{df}{d\xi}\right)_1 + 20\xi_s + 1\right] - \qquad (2.3)$$

$$\left[\left(\frac{df}{d\xi}\right)_2 + 20\xi_s + 2\right]\xi_s - 8\xi_s^2 = 0$$

In numerical computations of the shock front position and of the flow behind it we seek zeros of function $\Phi(\xi)$ which appears in the left-hand part of (2.3)

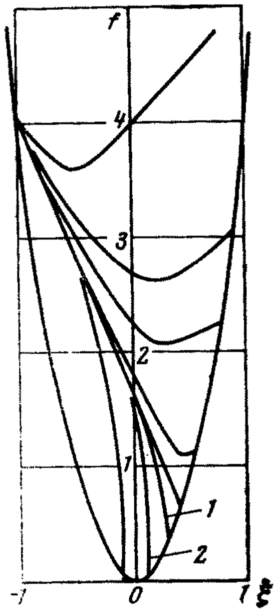

Fig. 3

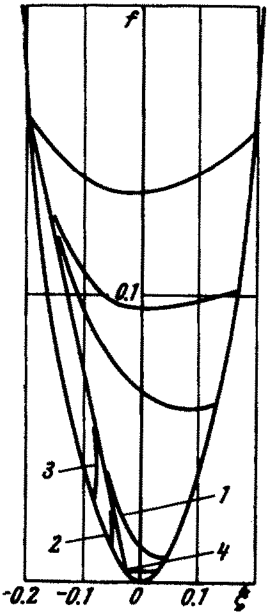

Fig. 4

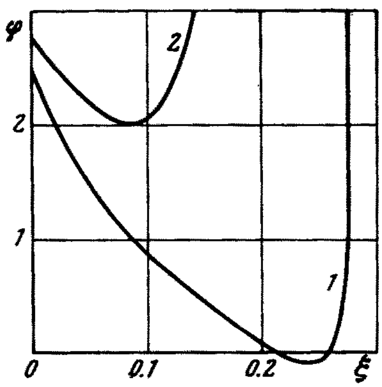

Fig. 5

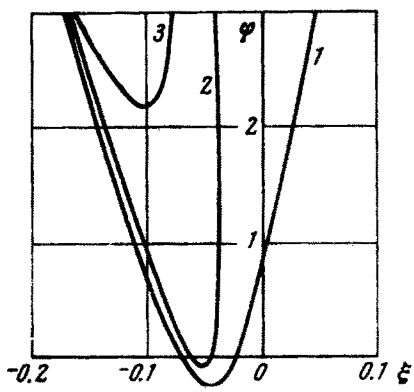

Fig. 6

Curves of $\Phi(\xi)$ corresponding to the integral curves appearing in Figs. 3 and 4 are shown in Figs. 5 and 6, respectively. Curves of $\Phi(\xi)$ denoted by numerals 1, 2 and 3 correspond to integral curves with the same numerals in the (f, ξ)-plane. A certain limit curve $\Phi(\xi)$ exists in every case; it vanishes at one point, and it is not possible to satisfy conditions (2.3) for all curves lying below it. Such flows cannot be presented in their entirety in the self-similar form indicated here.

It is seen from Fig. 6 that conditions (2.3) can be satisfied not only along the integral curves that define flows with a limit line but, also, along some curves for which continuous solutions exist. Furthermore, unlike in the case of inert gas, a shock front may be generated at coordinate $\xi_s < 0$. Flows with the shock wave reaching the nozzle center do not evidently obtain under real conditions. They correspond to flows in nozzles with wall discontinuities.

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ON PROPAGATION OF HEAT IN ONE-DIMENSIONAL DISPERSE MEDIA

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It is shown that solutions of the first boundary value problem for second order linear parabolic equation with two independent variables reduce in region ω with weak convergence of its coefficients in $L_2(\omega)$ to the solution of the first boundary value problem for some limit equation. This means that solution of the "microscopic" problem of heat propagation in one-dimensional disperse medium can be approximated by the solution of the "macroscopic" problem.

The basic problem of the theory of disperse media consists of the determination of macroscopic properties of these by the known properties of their constituents and by the macroscopic parameters which depend on the disperse medium structure. A strict mathematical formulation of this problem in a general form has not been so far achieved (see surveys [1, 2]). Statistical methods had been applied to the investigation of properties of disperse media [3 — 5]. Another approach consists in the analysis of equations with discontinuous coefficients that define disperse media at a "microscopic" level with the view to approximating solutions of such equations by functions which satisfy equations whose coefficients are in a certain sense limiting and possess better differential properties than the coefficients of input equations (see [6 — 8]). This problem has not yet been analyzed in a general form. Supplementary restrictions were imposed in the considered cases on the structure of coefficients of input equations, as for example, the condition of periodicity [9, 10] or of other kind [6, 11].